Constrained optimization of a production function

Given the following production function:

$$P(K, L) = 50K^{\beta}L$$

and the constraint:

$$4K + 15L = 100$$

a) If they exist, find the critical point(s) that optimize the production function under the given constraint.

b) Consider $\beta = 2$. Construct the bordered Hessian matrix and conclude whether it is a constrained maximum or minimum.

Solution

a)

Given the production function:

$$P(K,L) = 50K^{\beta}L$$

and the constraint:

$$4K + 15L = 100$$

To optimize $P(K,L)$ under the constraint, we use the method of Lagrange multipliers. Define the Lagrange multiplier λ and construct the Lagrangian function:

$$\mathcal{L}(K,L,\lambda) = 50K^{\beta}L + \lambda(100 - 4K - 15L)$$

Compute the partial derivatives of $\mathcal{L}$ with respect to K, L, and λ:

$$\frac{\partial \mathcal{L}}{\partial K} = 50\beta K^{\beta-1}L - 4\lambda = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial L} = 50K^{\beta} - 15\lambda = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 100 - 4K - 15L = 0 \quad (3)$$

From equation (2), solve for λ:

$$\lambda = \frac{50K^{\beta}}{15}$$

Substitute λ into equation (1):

$$50\beta K^{\beta-1}L - 4\left(\frac{50K^{\beta}}{15}\right) = 0$$

Simplify:

$$50\beta K^{\beta-1}L - \frac{200K^{\beta}}{15} = 0$$

$$\beta K^{\beta-1}L - \frac{4K^{\beta}}{15} = 0$$

$$15\beta K^{\beta-1}L - 4K^{\beta} = 0$$

Notice that $K^{\beta} = K^{\beta-1}K$, so factor $K^{\beta-1}$:

$$K^{\beta-1}(15\beta L - 4K) = 0$$

Assuming $K^{\beta-1} \neq 0$, then:

$$15\beta L - 4K = 0$$

Solve for K:

$$K = \frac{15\beta}{4}L$$

Substitute K into the constraint (3):

$$100 - 4\left(\frac{15\beta}{4}L\right) - 15L = 0$$

Simplify:

$$100 - 15\beta L - 15L = 0$$

$$100 - 15L(\beta + 1) = 0$$

Solve for L:

$$L = \frac{100}{15(\beta+1)} = \frac{20}{3(\beta+1)}$$

Substitute L into the expression for K:

$$K = \frac{15\beta}{4} \cdot \frac{20}{3(\beta+1)} = \frac{100\beta}{4(\beta+1)} = \frac{25\beta}{\beta+1}$$

The critical points that optimize the production function are:

$$L^* = \frac{20}{3(\beta+1)}, \quad K^* = \frac{25\beta}{\beta+1}$$

b)

With $\beta = 2$, the optimal values are:

$$L^* = \frac{20}{3(2+1)} = \frac{20}{9}, \quad K^* = \frac{25 \times 2}{2+1} = \frac{50}{3}$$

The Lagrangian function is:

$$\mathcal{L}(K, L, \lambda) = 50K^2L + \lambda(100 - 4K - 15L)$$

Compute the necessary second derivatives:

- First derivatives:

$$\frac{\partial \mathcal{L}}{\partial K} = 100KL - 4\lambda$$
$$\frac{\partial \mathcal{L}}{\partial L} = 50K^2 - 15\lambda$$

- Second derivatives:

$$\mathcal{L}_{KK} = \frac{\partial^2 \mathcal{L}}{\partial K^2} = 100L$$
$$\mathcal{L}_{LL} = \frac{\partial^2 \mathcal{L}}{\partial L^2} = 0$$
$$\mathcal{L}_{KL} = \mathcal{L}_{LK} = \frac{\partial^2 \mathcal{L}}{\partial K \partial L} = 100K$$

- Derivatives with respect to λ:

$$\frac{\partial g}{\partial K} = -4, \quad \frac{\partial g}{\partial L} = -15$$

The bordered Hessian matrix is:

$$H = \begin{pmatrix} 0 & \frac{\partial g}{\partial K} & \frac{\partial g}{\partial L} \\ \frac{\partial g}{\partial K} & \mathcal{L}_{KK} & \mathcal{L}_{KL} \\ \frac{\partial g}{\partial L} & \mathcal{L}_{LK} & \mathcal{L}_{LL} \end{pmatrix} = \begin{pmatrix} 0 & -4 & -15 \\ -4 & 100L & 100K \\ -15 & 100K & 0 \end{pmatrix}$$

Substitute the optimal values:

$$L^* = \frac{20}{9}, \quad K^* = \frac{50}{3}$$

Calculate:

$$100L^* = 100 \times \frac{20}{9} = \frac{2000}{9}$$

$$100K^* = 100 \times \frac{50}{3} = \frac{5000}{3}$$

The matrix evaluated at the critical point is:

$$H = \begin{pmatrix} 0 & -4 & -15 \\ -4 & \frac{2000}{9} & \frac{5000}{3} \\ -15 & \frac{5000}{3} & 0 \end{pmatrix}$$

Compute the determinant of H:

$$\det(H) = \begin{vmatrix} 0 & -4 & -15 \\ -4 & \frac{2000}{9} & \frac{5000}{3} \\ -15 & \frac{5000}{3} & 0 \end{vmatrix}$$

Using the method of cofactors:

$$\det(H) = -(-4) \begin{vmatrix} \frac{2000}{9} & \frac{5000}{3} \\ \frac{5000}{3} & 0 \end{vmatrix} - (-15) \begin{vmatrix} -4 & \frac{5000}{3} \\ -15 & 0 \end{vmatrix}$$

$$= 4 \left(\frac{2000}{9} \times 0 - \frac{5000}{3} \times \frac{5000}{3} \right) + 15 \left(-4 \times 0 - (-15) \times \frac{5000}{3} \right)$$

Calculate the terms:

$$\text{First minor: } \frac{2000}{9} \times 0 - \frac{5000}{3} \times \frac{5000}{3} = -\left(\frac{5000}{3}\right)^2 = -\frac{(5000)^2}{9}$$

$$\text{Second minor: } -4 \times 0 - (-15) \times \frac{5000}{3} = \frac{15 \times 5000}{3} = \frac{75000}{3} = 25000$$

Therefore:

$$\det(H) = 4 \left(-\frac{(5000)^2}{9} \right) + 15 \times 25000$$

$$= -\frac{4 \times 25,000,000}{9} + 375,000$$

$$= -\frac{100,000,000}{9} + 375,000$$

Simplify:

$$-\frac{100,000,000}{9} + \frac{3,375,000}{9} = -\frac{96,625,000}{9}$$

Finally, we obtain:

$$\det(H) = -\frac{96,625,000}{9}$$

We observe that $\det(H) < 0$.

Since the determinant of the bordered Hessian matrix is negative, and considering that we are seeking a constrained maximum (since it is a production function), we conclude that it is a constrained maximum at the critical point found.